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ARTICLE

On The Change of Form of Long Waves Advancing in a Rectangular Canal, and on a New Type of Long Stationary Waves

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1. Introduction

In such excellent treatises on hydrodynamics as those of Lamb, [6], and Basset, [2], we find that even when friction is neglected, long waves in a rectangular canal must necessarily change their form as they advance, becoming steeper in front and less steep behind[1]. Yet since the investigations of de Boussinesq ([3]), Lord Rayleigh ([8]), and St. Venant ([9]) on the solitary wave, there has been some cause to doubt the truth of this assertion. Indeed, if the reasons adduced were really decisive, it is difficult to see why the solitary wave should make an exception[2]; but even Lord Rayleigh and McCowan ([7]), who have successfully and thoroughly treated the theory of this wave, do not directly contradict the statement in question. They are, as it seems to us, inclined to the opinion that the solitary wave is only stationary to a certain approximation. It is the desire to settle this question definitively which has led us into the somewhat tedious calculations which are to be found at the end of our paper. We believe, indeed, that from them the conclusion may be drawn, that in a frictionless liquid there may exist absolutely stationary waves and that the form of their surface and the motion of the liquid below it may be expressed by means of rapidly convergent series. But, in order that these lengthy calculations might not obscure other results, which were obtained in a less elaborate way, we have postponed them to the last part of our paper. First, then, we investigate the deformation of a system of waves of arbitrary shape but moving in one direction only, i.e. we consider one of the two systems of waves, starting in opposite directions in consequence of any disturbance, after their complete separation from each other. By adding to the motion of the fluid a uniform motion with velocity equal and opposite to the velocity of propagation of the waves, we may reduce the surface of such a system to approximate, but not perfect, rest.

If, then, $l + \eta$ (η being a small quantity) represent the elevation of the surface above the bottom at a horizontal distance x from the origin of coordinates, we have succeeded in deducing the equation

$$\boxed{\frac{\partial \eta}{\partial t} = \frac{3}{2}\sqrt{\frac{g}{l}} \cdot \frac{\partial\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2\eta}{\partial x^2}\right)}{\partial x}}$$

---

[1] It seems that this opinion was expressed for the first time by Airy, [1]

[2] Though the theory of the solitary wave is duly discussed in the treatise of Basset, the inconsistency of his result with the doctrine of the necessary change of form of long waves seems not to have sufficiently attracted the attention of the author.

where α is a small but arbitrary constant, which is in close connexion with the exact velocity of the uniform motion given to the liquid, and where $\sigma = \frac{1}{3}l^3 - \frac{Tl}{\rho g}$ depends upon the depth l of the liquid, upon the capillary tension T at its surface, and upon its density ρ.
On assuming $\frac{\partial \eta}{\partial t} = 0$ we of course obtain the differential equation for stationary waves, and it is easily shown that the well-known equation

$$\eta = h \cdot \operatorname{sech}^2 x \cdot \sqrt{\frac{h}{4\sigma}}$$

of the solitary wave is included as a particular case in the general solution of this equation[3]. But, in referring to this kind of wave, we have to notice the result that, taking capillarity into account, a negative wave will become the stationary one, when the depth of the liquid is small enough. On proceeding then to the general solution, a new type of long stationary wave is detected, the shape of the surface being determined by the equation

$$\eta = h \cdot \operatorname{cn}^2 x \cdot \sqrt{\frac{h+k}{4\sigma}} \quad \left(\operatorname{mod} \mathbf{M} = \sqrt{\frac{h}{h+k}}\right)$$

where h is a constant and k with $0 < k < 1$ is the elliptic modulus of the function cn, known as cosinus amplitudinis.
We propose to attach to this type of wave the name of *cnoidal* waves (in analogy with sinusoidal waves). For $k = 0$[4] they become identical with the solitary wave. For large values of k they bear more and more resemblance to sinusoidal waves, though their general aspect differs in this respect, that their elevations are narrower than their hollows; at least when the liquid is not too shallow, in which latter case this peculiar feature is reversed by the influence of capillarity. For very large values of k these cnoidal waves coincide with the train of oscillatory waves of unchanging shape discovered by Stokes ([13]), which therefore *in the theory of long waves*[5] constitutes a particular case of the cnoidal form. Indeed the equation ([12], page

---

[3] The functions secans hyperbolicus or hyperbolic secant and cosecans hyperbolicus or hyperbolic cosecant are hyperbolic functions and are defined according to:

$$\operatorname{sech}\ x = \frac{2}{e^x + e^{-x}} = \frac{1}{\cosh x}$$
$$\operatorname{csch}\ x = \frac{2}{e^x - e^{-x}} = \frac{1}{\sinh x}$$

They are per definition the inverse of hyperbolic sine and hyperbolic cosine.

[4] see Appendix B, Table B.1

[5] Stokes solution is more general in so far as it applies also to those eases wherein the depth of the liquid is moderate or large in respect to the wave-length.

210) obtained by Stokes, when written in our notation, becomes

$$\eta = h \cos\left(\frac{2\pi x}{\lambda}\right) - \frac{3h^2\lambda^2}{64\pi^2 l^3} \cdot \cos\left(\frac{4\pi x}{\lambda}\right)$$

but, as Sir G. Stokes remarks, in order that the method of approximation adopted by him may be legitimate, $\frac{\lambda^2 h}{l^3}$ must be a small fraction. Now, when capillarity is neglected, the wave-length λ of our cnoidal waves is equal to

$$\frac{4K \cdot \sqrt{l^3}}{\sqrt{3 \cdot (h+k)}} \,,$$

with $K = K(k)$ being the complete elliptic integral of the first kind and therefore

$$\frac{\lambda^2 h}{l^3} = \frac{16K^2 h}{3 \cdot (h+k)} = \frac{16}{3} \mathbf{M}^2 K^2 \,.$$

This is a small fraction only when $\mathbf{M}$, the modulus, is small, but the cnoidal waves then resemble sinusoidal waves; and it is obvious that in this case the equation of their surface may be developed in a rapidly convergent Fourier-series, of which Sir G. Stokes has given the first two terms.

After some more discussion about these cnoidal waves, concerning their velocity of propagation and the motion of the particles of fluid below their surface, we proceed to a closer examination of the deformation of long waves. To this effect we apply the equation for $\frac{\partial \eta}{\partial t}$ to various types of non-stationary waves, and it will appear that, though sinusoidal waves become steeper in front when advancing, other types of waves may behave otherwise.

2. The formula for $\frac{\mathrm{d}\eta}{\mathrm{d}t}$

In our investigations (in accordance with the method used by Lord Rayleigh, ([8], page 257, whose paper has been of great influence on our researches) we start from the supposition that the horizontal and vertical velocities u and v of the fluid may be expressed by rapidly convergent series of the form

$$\begin{aligned} u &= f + yf_1 + y^2 f_2 + \cdots \\ v &= y\phi_1 + y^2\phi_2 + \cdots \end{aligned}$$

where y represents the height of a particle above the bottom of the canal, and where f, $f_1, \ldots$, ϕ_1, $\phi_2, \ldots$ are functions of the horizontal distance x and time t. Of course the validity of this assumption must be proved later on by the fact that series of this description can be found satisfying all the conditions of the problem. From one of these conditions, viz., the incompressibility of the liquid, which is expressed by $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, we may deduce

$$\phi_n = -\frac{1}{n} \cdot \frac{\partial f_{n-1}}{\partial x}\,,$$

and from another, viz., the absence of rotation in the fluid expressed by $\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$:

$$\begin{aligned} f_1 &= 0 \\ f_n &= \frac{1}{n} \cdot \frac{\partial \phi_{n-1}}{\partial x} = -\frac{1}{n \cdot (n-1)} \cdot \frac{\partial^2 f_{n-2}}{\partial x^2}\,. \end{aligned}$$

In this manner we obtain the following set of equations:

$$u = f - \frac{1}{2}y^2 \cdot \frac{\partial^2 f}{\partial x^2} + \frac{1}{24}y^4 \cdot \frac{\partial^4 f}{\partial x^4} - \cdots \tag{2.1}$$

$$v = -y \cdot \frac{\partial f}{\partial x} + \frac{1}{6}y^3 \cdot \frac{\partial^3 f}{\partial x^3} - \frac{1}{120}y^5 \cdot \frac{\partial^5 f}{\partial x^5} + \cdots\,, \tag{2.2}$$

and, moreover, if ϕ be the velocity potential and ψ the streamfunction:

$$\phi = \int f\partial x - \frac{1}{2}y^2 \cdot \frac{\partial f}{\partial x} + \frac{1}{24}y^4 \cdot \frac{\partial^3 f}{\partial x^3} - \cdots \tag{2.3}$$

$$\psi = y \cdot f - \frac{1}{6}y^3 \cdot \frac{\partial^2 f}{\partial x^2} + \frac{1}{120}y^5 \cdot \frac{\partial^4 f}{\partial x^4} - \cdots \tag{2.4}$$

which set of equations satisfies for the interior of the fluid all the conditions of the problem, whilst at the same time it is easy to see that for long waves these series are rapidly convergent. Indeed, for such waves the state of motion changes slowly with x, and therefore the successive differential quotients with respect to this variable of all functions referring, as f does, to the state of motion, must rapidly decrease. Passing now to the conditions at the boundary, let p_1 (a constant) be the atmospheric pressure, p_1' the pressure at a point below the surface where the capillary forces cease to act, and T the surface-tension. We then have, distinguishing here and elsewhere by the suffix ($_1$) those quantities which refer to the surface,

$$p_1' = p_1 - T \cdot \frac{\partial^2 y_1}{\partial x^2}\,;$$

but according to a well-known equation of hydrodynamics[1],

$$\frac{p_1'}{\rho} = \chi(t) - \frac{\partial \phi_1}{\partial t} - \frac{1}{2} \cdot \left(u_1^2 + v_1^2\right) - g y_1\,,$$

therefore

$$\begin{aligned} \frac{p_1'}{\rho} &= \chi(t) - \frac{\mathrm{d}\phi_1}{\mathrm{d}t} - \frac{1}{2} \cdot \left(u_1^2 + v_1^2\right) - g y_1 + \frac{T}{\rho} \cdot \frac{\partial^2 y_1}{\partial x^2} \\ &= L - g y_1 + M y_1^2 + N y_1^4 + P y_1^6 + \cdots + \frac{T}{\rho} \cdot \frac{\partial^2 y_1}{\partial x^2}\,, \end{aligned} \tag{2.5}$$

---

[1] The BERNOULLI equation, which reads in the case of unsteady potential flow:

$$\frac{\partial \phi}{\partial t} + \tfrac{1}{2}v^2 + \frac{p}{\rho} + g y = \chi(t)\,,$$

valid also for unsteady—or time dependent—flows. Here $\frac{\partial \phi}{\partial t}$ denotes the partial derivative of the velocity potential ϕ with respect to time t, and $v = |\nabla \phi|$ is the flow speed. The function $\chi(t)$ depends only on time and not on position in the fluid.

where

$$
\begin{aligned}
L &= \chi(t) - \int \frac{\partial f}{\partial t}\,\mathrm{d}x - \frac{1}{2}\cdot f^2\,,\\
M &= \frac{1}{2}\cdot f\frac{\partial^2 f}{\partial x^2} + \frac{1}{2}\cdot\frac{\partial^2 f}{\partial x\,\partial t} - \frac{1}{2}\cdot\left(\frac{\partial f}{\partial x}\right)^2\,,\\
N &= -\frac{1}{24}\cdot f\frac{\partial^4 f}{\partial x^4} - \frac{1}{8}\cdot\left(\frac{\partial^2 f}{\partial x^2}\right) + \frac{1}{6}\cdot\frac{\partial f}{\partial x}\cdot\frac{\partial^3 f}{\partial x^3} - \frac{1}{24}\cdot\frac{\partial^4 f}{\partial x^3\,\partial t}\,,\\
P &= \frac{1}{720}\cdot f\frac{\partial^6 f}{\partial x^6} + \frac{1}{48}\cdot\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^4 f}{\partial x^4} - \frac{1}{72}\cdot\left(\frac{\partial^3 f}{\partial x^3}\right)^2 - \frac{1}{120}\cdot\frac{\partial f}{\partial x}\cdot\frac{\partial^5 f}{\partial x^5} + \frac{1}{720}\cdot\frac{\partial^6 f}{\partial x^5\,\partial t}\,.
\end{aligned}
$$

By differentiation with respect to x, Equation 2.5 may be written as

$$
\begin{aligned}
\frac{\partial L}{\partial x} + y_1^2\cdot\frac{\partial M}{\partial x} + y_1^4\cdot\frac{\partial N}{\partial x} + y_1^6\cdot\frac{\partial P}{\partial x} + \cdots + g\cdot\frac{\partial y_1}{\partial x} + 2My_1\cdot\frac{\partial y_1}{\partial x}&\\
+ 4Ny_1^3\cdot\frac{\partial y_1}{\partial x} + 6Py_1^5\cdot\frac{\partial y_1}{\partial x} + \cdots + \frac{T}{\rho}\cdot\frac{\partial^3 y_1}{\partial x^3} = 0\,.& \qquad (2.6)
\end{aligned}
$$

Moreover, a second equation must hold good at the surface, viz.

$$
-u_1\cdot\frac{\partial y_1}{\partial x} + v_1 - \frac{\partial y_1}{\partial t} = 0\,. \qquad (2.7)
$$

In order to satisfy Equation 2.6 and Equation 2.7 by the method of successive approximations, we put $y_1 = l + \eta$, $f = q_0 + \beta$, where l and q_0 are supposed to be constants, and η and β small functions depending upon x and t. Dealing then, with the fact that for long waves, whose wave-length is great in comparison with the depth of the canal, every new differentiation with respect to x gives rise to continually smaller quantities, these equations become as a *first* approximation:

$$
\begin{aligned}
q_0\cdot\frac{\partial\beta}{\partial x} + \frac{\partial\beta}{\partial t} + g\cdot\frac{\partial\eta}{\partial x} &= 0\,,\\
q_0\cdot\frac{\partial\eta}{\partial x} + \frac{\partial\eta}{\partial t} + l\cdot\frac{\partial\beta}{\partial x} &= 0\,,
\end{aligned}
$$

and are satisfied by taking

$$
\begin{aligned}
\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{\mathrm{d}\beta}{\mathrm{d}t} &= 0\,,\\
\beta &= -\frac{q_0}{l}\cdot(\eta + \alpha)\,,
\end{aligned}
$$

and

$$
q_0 = \sqrt{gl}\,, \qquad (2.8)
$$

where α is an arbitrary constant which we will suppose to be small.

It is obvious that this solution coincides with the one usually given for the case of

long waves of arbitrary shape made stationary by attributing to the fluid a velocity equal and opposite to that of the waves, on the assumption that the velocity in a vertical direction (v) may be neglected and that the horizontal velocity (u) may be considered uniform across each section of the canal.
But, if we wish to proceed to a *second* approximation, we have to put

$$f = q_0 - \frac{q_0}{l} \cdot (\eta + \alpha + \gamma)\,, \tag{2.9}$$

where γ is small compared with η and α. On substituting this in Equation 2.6 and Equation 2.7 and on writing out the result, rejecting all terms[2] which are small compared with any one of the remaining terms, we find respectively:

$$\frac{q_0}{l} \cdot \frac{\partial \eta}{\partial t} + g \cdot \frac{\partial \gamma}{\partial x} - \frac{g}{l} \cdot (\eta + \alpha) \cdot \frac{\partial \eta}{\partial x} - \left(\frac{1}{2} l^2 g - \frac{T}{\rho}\right) \frac{\partial^3 \eta}{\partial x^3} = 0\,, \tag{2.10}$$

and

$$\frac{q_0}{l} \cdot \frac{\partial \eta}{\partial t} - g \cdot \frac{\partial \gamma}{\partial x} - \frac{g}{l} \cdot (2\eta + \alpha) \cdot \frac{\partial \eta}{\partial x} + \frac{1}{6} l^2 g \cdot \frac{\partial^3 \eta}{\partial x^3} = 0\,. \tag{2.11}$$

In eliminating $\frac{\partial \gamma}{\partial x}$ from these equations, we have at last

$$\boxed{\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{3q_0}{2l} \cdot \frac{\partial \left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma \frac{\partial^2 \eta}{\partial x^2}\right)}{\partial x}} \tag{2.12}$$

where

$$\sigma = \frac{1}{3} \cdot l^3 - \frac{Tl}{\rho g}\,. \tag{2.13}$$

This very important equation, to which we shall have frequently to revert in the course of this paper, indicates the deformation of a system of waves of arbitrary shape, but moving in one direction only. Before applying it, we may point out the close connexion between the constant α, which may still be chosen arbitrarily, and the uniform velocity given to the fluid. Indeed it is easy to see from Equation 2.1 and Equation 2.9 how a variation $\delta\alpha$ of the constant α corresponds to a change $\delta q = \frac{q_0}{l}\delta\alpha$ in this velocity, but, on taking the variation of Equation 2.12 with respect to α, we obtain

$$\delta \cdot \frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{q_0}{l} \cdot \delta\alpha \cdot \frac{\partial \eta}{\partial x} = -\delta q \cdot \frac{\partial \eta}{\partial x}\,,$$

which equation may be easily verified geometrically.

---

[2] The term for instance with $\frac{\partial \eta}{\partial x}$, $\frac{\partial^2 \eta}{\partial x^2}$ and $\left(\frac{\partial \eta}{\partial x}\right)^3$ are rejected in comparison with $\eta \frac{\partial \eta}{\partial x}$, which is retained in the equation, those with $\frac{\partial \gamma}{\partial t}$ and $\frac{\partial^3 \eta}{\partial x^2 \partial t}$ against $\frac{\mathrm{d}\eta}{\mathrm{d}t}$.

3. Stationary Waves

For stationary waves, $\frac{\mathrm{d}\eta}{\mathrm{d}t}$ must be zero. Therefore we have from Equation 2.12

$$\partial\left(\frac{1}{2}\eta^2+\frac{2}{3}\alpha\eta+\frac{1}{3}\sigma\cdot\frac{\partial^2\eta}{\partial x^2}\right)=0\,.$$

This gives by integration

$$c_1+\frac{1}{2}\eta^2+\frac{2}{3}\alpha\eta+\frac{1}{3}\sigma\cdot\frac{\partial^2\eta}{\partial x^2}=0\,; \tag{3.1}$$

and by multiplication with $6\,\mathrm{d}\eta$ and further integration

$$c_2+6c_1\eta+\eta^3+2\alpha\eta^2+\sigma\cdot\left(\frac{\partial\eta}{\partial x}\right)^2=0\,. \tag{3.2}$$

If now the fluid be undisturbed at infinity, and if l be taken equal to the depth which it has there, then Equation 3.1 and Equation 3.2 must be satisfied by $\eta=0$, $\frac{\partial\eta}{\partial x}=0$, and $\frac{\partial^2\eta}{\partial x^2}=0$. Therefore, in *this* case c_1 and c_2 are equal to zero, and Equation 3.2 leads to

$$\frac{\partial\eta}{\partial x}=\pm\sqrt{-\frac{\eta^2\cdot(\eta+2\alpha)}{\sigma}}\,. \tag{3.3}$$

Here, before we can proceed, we have to discriminate between σ positive and σ negative. In the *first case* 2α is necessarily negative because $\frac{\partial\eta}{\partial x}$ must be real for small values of η. If then, we put it equal to $-h$, we have

$$\frac{\mathrm{d}\eta}{\mathrm{d}x}=\pm\sqrt{\frac{1}{\sigma}}\cdot\eta\cdot\sqrt{h-\eta}\,,$$

from which, supposing x to be zero for $\eta=h$, we easily obtain the well-known equation of the positive solitary wave, viz.:

$$\eta=h\cdot\operatorname{sech}^2 x\cdot\sqrt{\frac{h}{4\sigma}}\,. \tag{3.4}$$

In the *second case* 2α must be positive. In putting it equal to h, and in substituting $-\eta'$ for η, we have from Equation 3.3

$$\frac{\partial \eta'}{\partial x} = \pm\sqrt{-\frac{1}{\sigma}} \cdot \eta' \cdot \sqrt{h-\eta'}\,,$$

or, by integration,

$$\eta = -\eta' = -h \cdot \operatorname{sech}^2 x \cdot \sqrt{-\frac{h}{4\sigma}}\,.$$

This is the equation of a *negative* solitary wave, and we are able now to draw the conclusion that whenever σ is negative; that is whenever the depth of the liquid is less than $\sqrt{\frac{3T}{g\rho}}$, the stationary wave is a negative one. For water at 20 °C this limiting depth is equal to 0.47 cm ($T = 72$ dyn/cm , $g = 981\,\mathrm{cm}^2\,\mathrm{s}^{-1}$, $\rho = 0.998$ B.A.U.).
Now, for a further discussion of Equation 3.2, we drop the assumption that the fluid is undisturbed at infinity. If then l be taken equal to the smallest depth of the liquid, we must have $\frac{\partial \eta}{\partial x} = 0$ for $\eta = 0$, and therefore in virtue of Equation 3.2 $c_2 = 0$. On supposing then σ positive[1], c_1 must be negative in order that $\frac{\partial \eta}{\partial x}$ may be real for small positive values of η, but then the equation

$$\eta^2 + 2\alpha\eta + 6c_1 = 0 \tag{3.5}$$

has a positive root h and a negative $-k$, and we may get from Equation 3.2

$$\frac{\partial \eta}{\partial x} = \pm\sqrt{\frac{1}{\sigma}\eta \cdot (h-\eta) \cdot (k+\eta)}\,. \tag{3.6}$$

By substitution in this Equation 3.6 of $\eta = h\cos^2\chi$ and by integration, we find

$$\eta = h \cdot \operatorname{cn}^2 x \cdot \sqrt{\frac{h+k}{4\sigma}} \left(\operatorname{mod}\mathbf{M} = \sqrt{\frac{h}{h+k}}\right)\,, \tag{3.7}$$

which is the equation of a train of periodic waves whose wave-length increases when k decreases.

---

[1] When σ is negative, let then l be equal to the *greatest* depth. On substituting $\sigma = -\sigma'$, $\eta = -\eta'$ we have again c_1 negative,

$$\left(\frac{\mathrm{d}\eta'}{\mathrm{d}x}\right)^2 = \frac{1}{\sigma'}\eta' \cdot (h-\eta') \cdot (k+\eta')\,,$$

and finally

$$\eta = -\eta' = -h \cdot \operatorname{cn}^2 x \cdot \sqrt{\frac{h+k}{4\sigma'}}\,,$$

where h and $-k$ are the roots of the quadratic equation $\eta'^2 - 2\alpha\eta + 6c_1 = 0$

For $k = 0$ this length becomes infinite, and the equation may be shown to coincide with Equation 3.4. The following Figure 3.1 represents such a train of stationary waves for the case in which $k = \frac{9}{16}h$, $\mathbf{M} = 0.8$.

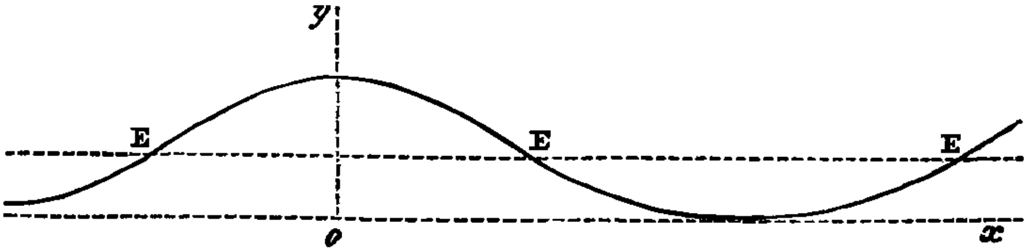

Figure 3.1.: Train of stationary waves

4. Stationary Periodic Waves (Cnoidal Waves)

Proceeding now to a further investigation of the waves determined by Equation 3.7, we calculate from Equation 2.10 and Equation 2.11 the value of γ. From these equations we get

$$\frac{\mathrm{d}\gamma}{\mathrm{d}x} = -\frac{1}{2l}\eta \cdot \frac{\partial \eta}{\partial x} + \left(\frac{1}{3}l^2 - \frac{T}{2g\rho}\right) \cdot \frac{\partial^3 \eta}{\partial x^3}\,,$$

or, by integration

$$\gamma = -\frac{1}{4l}\eta^2 + \left(\frac{1}{3}l^2 - \frac{T}{2g\rho}\right) \cdot \frac{\partial^2 \eta}{\partial x^2}\,,$$

where the constant of integration is rejected because its retention would only have had the effect of augmenting in Equation 2.9 the value of the arbitrary constant α. On substituting, then, f from Equation 2.9 in Equation 2.1 and Equation 2.2, observing that in virtue of Equation 3.1

$$\frac{\partial^2 \eta}{\partial x^2} = -\frac{1}{2\sigma} \cdot \left(3\eta^2 + 4\alpha\eta + 6c_1\right) = -\frac{1}{2\sigma} \cdot \left(3\eta^2 - 2 \cdot (h-k) \cdot \eta - hk\right)\,,$$

these equations are replaced by

$$\begin{aligned} u = \sqrt{gl} - \sqrt{\frac{g}{l}} \cdot \Bigg\{\eta + \frac{1}{2} \cdot (k-h) - \frac{\eta^2}{4l} + \left(\frac{1}{l} + \frac{T}{2g\rho\sigma}\right) \cdot \Bigg[(h-k) \cdot \eta \\ + \frac{1}{2}kh - \frac{3}{2}\eta^2\Bigg]\Bigg\} + \frac{1}{2\sigma} \cdot \sqrt{\frac{g}{l}} \cdot \left\{(h-k) \cdot \eta + \frac{1}{2}hk - \frac{3}{2}\eta^2\right\} \cdot y^2 + \cdots \end{aligned} \tag{4.1}$$

$$v = \sqrt{\frac{g\eta \cdot (h-\eta) \cdot (k+\eta)}{l\sigma}} \cdot y\,. \tag{4.2}$$

When $k = 0$, they determine the motion of the fluid for a solitary wave.

In the first place we now will endeavour to calculate the velocity of propagation. For the solitary wave this is simple enough. If we consider that the liquid at infinity is brought to rest when a uniform motion with a horizontal velocity

$$-q = -\sqrt{gl} \cdot \left(1 + \frac{h}{2l}\right) \tag{4.3}$$

is added to the motion expressed by Equation 4.1 and Equation 4.2, it is clear that this velocity, with reversed sign, must be taken for the velocity of propagation of the solitary wave.
But for a train of oscillatory waves Sir G. Stokes has shown ([12] ,page 202) that various definitions of this velocity may be given, leading at the higher order of approximation to different values. It seemed to us most rational to define it as the velocity of propagation of the wave-form when the *horizontal momentum* of the liquid has been reduced to zero by the addition of a uniform motion. This definition corresponds to the second one of Sir G. Stokes. According to it, we have to solve the equation

$$\int_0^\lambda \mathrm{d}x \int_0^{l+\eta} (u - q)\,\mathrm{d}y = 0\,, \tag{4.4}$$

where q denotes the velocity of propagation, and where

$$\lambda = \frac{2K\sqrt{\sigma}}{\sqrt{h+k}} \tag{4.5}$$

is equal to the wave-length.
If, then,

$$\begin{aligned} V = \int_0^\lambda \eta\,\mathrm{d}x &= 4\sqrt{\frac{\sigma}{h+k}} \cdot \{(h+k)\cdot E(K) - kK\} \\ &= \lambda \cdot \left\{(h+k)\cdot \frac{E(K)}{K} - k\right\} \end{aligned} \tag{4.6}$$

denote the volume of a single wave reckoned from above its lowest point, we get from Equation 4.4, retaining only such terms as are of the first order compared with η, h, and k:

$$\begin{aligned} q &= \frac{\int_0^\lambda \mathrm{d}x \int_0^{l+\eta} u \,\mathrm{d}y}{\int_0^\lambda \mathrm{d}x \int_0^{l+\eta} \mathrm{d}y} = \frac{\int_0^\lambda \left\{\sqrt{gl} - \sqrt{\frac{g}{l}}\eta - \frac{1}{2} \cdot (k-h) \cdot \sqrt{\frac{g}{l}}\right\} \cdot (l+\eta)\,\mathrm{d}x}{\int_0^\lambda (l+\eta)\,\mathrm{d}x} \\ &= \frac{\sqrt{gl} \cdot \left(1 - \frac{(k-h)}{2l}\right) \cdot l\lambda}{l\lambda + V} \\ &= \sqrt{gl} \cdot \left(1 - \frac{(k-h)}{2l} - \frac{V}{l\lambda}\right) \\ &= \sqrt{gl} \cdot \left(1 + \frac{(k+h)}{2l} - \frac{(k+h)}{l} \cdot \frac{E(K)}{K}\right) . \end{aligned} \tag{4.7}$$

On subtracting this velocity from that expressed by Equation 4.1, we obtain

$$\begin{aligned} u' = u - q &= -\sqrt{\frac{g}{l}} \cdot \left(\eta + k - (k+h) \cdot \frac{E(k)}{K}\right) \\ &= -\sqrt{\frac{g}{l}} \cdot \left(\eta - \frac{V}{l}\right) \end{aligned} \tag{4.8}$$

and it is obvious at once that in this manner we have annulled the velocity of the particles for which

$$\eta = \frac{V}{l} .$$

This last equation has a simple geometrical meaning. It designates those particles E (Figure 3.1) whose height above the bottom of the channel is equal to the height where the surface of the liquid would stand when the waves were flattened. Therefore, for a first approximation we may say that the various particles of the fluid change the direction of their horizontal motion at the very moment when one of these points E is passing over them.

We now proceed to the calculation of the path of a single particle of fluid. Let x_0 and y_0 denote the coordinates of such a particle at the origin of time, and $x' = x_0 + \xi'$, $y' = y_0 + \zeta'$ its coordinates at the time t, u' and v' its horizontal and vertical velocity at that time, $l + \eta'$ its elevation above the bottom, then we have

$$\begin{aligned} \xi' &= \int_0^t u' \,\mathrm{d}t = \sqrt{\frac{g}{l}} \cdot \int_0^t \left(\eta' - \frac{V}{\lambda}\right) \mathrm{d}t \\ \zeta' &= \int_0^t v' \,\mathrm{d}t = \sqrt{\frac{g}{l}} \cdot \int_0^t y \cdot \frac{\partial \eta'}{\partial x'} \,\mathrm{d}t . \end{aligned}$$

Here η' is equal to the value of η for $x = x' + qt$; and therefore we have $\mathrm{d}x = (u' - q)\,\mathrm{d}t$, or to a first approximation $\mathrm{d}t = -\frac{1}{q}\,\mathrm{d}x = -\frac{1}{\sqrt{gl}}\,\mathrm{d}x$; but then

$$\xi' = -\frac{1}{l} \cdot \int_{x_0}^{x_0+\sqrt{gl}\cdot t} \left(\eta - \frac{V}{\lambda}\right) \mathrm{d}x = \frac{V \cdot t}{\lambda}\sqrt{\frac{g}{l}} - \frac{h}{l} \cdot \int_{x_0}^{x_0+\sqrt{gl}\cdot t} \operatorname{cn}^2 \frac{2Kx}{\lambda}\,\mathrm{d}x\,.$$

Or, according to a well-known formula[1] (see [4], ch. VI, §187, page 143)

$$\xi' = \frac{(h+k)\cdot\lambda}{2Kl} \cdot \left[Z \cdot \left(\frac{2K \cdot \left(x_0 + \sqrt{gl}\cdot t\right)}{\lambda} \right) - Z \cdot \left(\frac{2Kx_0}{\lambda} \right) \right]. \tag{4.9}$$

At the same time we have

$$\begin{aligned}\xi' = -\frac{1}{l} \cdot y \cdot \int_{x_0}^{x_0+\sqrt{gl}\cdot t} \frac{\partial \eta}{\partial x}\,\mathrm{d}x = -\frac{h}{l} \cdot y \cdot \Bigg[&\operatorname{cn}^2 \left(x_0 + \sqrt{gl}\cdot t\right) \cdot \sqrt{\frac{(h+k)}{4\sigma}} \\ &- \operatorname{cn}^2 x_0 \cdot \sqrt{\frac{(h+k)}{4\sigma}} \Bigg]. \end{aligned} \tag{4.10}$$

Of course, as all fluid particles with the same y describe congruent paths, these formulæ may be simplified by supposing $x_0 = 0$.

---

[1] $Z(u) = u\left(1 - \frac{E(K)}{K}\right) - \mathbf{M}^2 \int_0^u \operatorname{sn} u\,\mathrm{d}u$

5. Deformation of Non-Stationary waves

In order to study the deformation of non-stationary waves, we will now apply our Equation 2.12 to various types of waves.

5.1. Solitary waves

As a first example we choose a solitary wave whose surface is given by

$$\eta = h \cdot \operatorname{sech}^2 px \tag{5.1}$$

According to Equation 2.12 the deformation of this wave is expressed by

$$\begin{aligned}\frac{\mathrm{d}\eta}{\mathrm{d}t} = -\frac{3q_0ph}{l} \cdot \bigl(4\sigma p^2 - h\bigr) \cdot \biggl[&- \operatorname{sech}^2 px \\ \frac{2}{3} \cdot \frac{(\alpha + 2\sigma p^2)}{(4\sigma p^2 - h)} \biggr] &\cdot \operatorname{sech}^2 px \cdot \tanh^3 px \,.\end{aligned} \tag{5.2}$$

But before we are able to draw any conclusion from this expression, it is necessary to separate the two parts of $\frac{\mathrm{d}\eta}{\mathrm{d}t}$, of which the first is due to a true change of form of the wave-surface, whilst the second may be attributed to a small advancing motion of the wave, which is left after the addition of the uniform motion with velocity $q_0 = \sqrt{gl}$. To this effect we have still at our disposal the quantity α, whose close connexion with the uniform motion, which we have added in order to make the wave nearly stationary, has been indicated above.

One of the best ways to obtain the desired separation is certainly to make stationary the highest point of the wave, and this is effected by fulfilling the condition

$$2 \cdot \bigl(\alpha + 2\sigma p^2\bigr) = 3 \cdot \bigl(4\sigma p^2 - h\bigr) \ ,$$

or

$$\alpha = 4\sigma p^2 - \frac{3}{2}h \,;$$

for in that case Equation 5.2 is simplified to

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = -\frac{3q_0ph}{l} \cdot \bigl(4\sigma p^2 - h\bigr) \cdot \operatorname{sech}^2 px \cdot \tanh^3 px \tag{5.3}$$

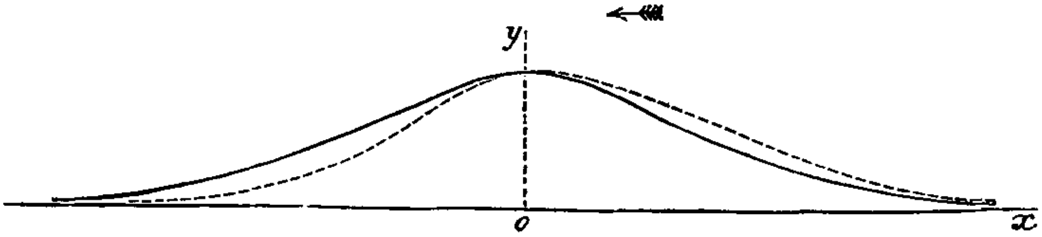

Figure 5.1.: Change of form of a wave from Equation 5.3

and then, for $x = 0$,

$$\frac{\partial \frac{\partial \eta}{\partial x}}{\partial t} = \frac{\partial \frac{\partial \eta}{\partial t}}{\partial x} \quad \text{is zero together with} \quad \frac{\partial \eta}{\partial x}\,.$$

In discussing this Equation 5.3, we see at once that a solitary wave according to Equation 5.1 is stationary when $h = 4\sigma p^2$; and this is in accordance with the Equation 3.4 of the stationary solitary wave which we have obtained above. When $h > 4\sigma p^2$, the change of form of the wave, calculated from Equation 5.3, is shown by the dotted line in Figure 5.1. Here the wave becomes steeper in front[1], whilst for $h < 4\sigma p^2$ the figure would show the opposite change of form, when, contrary to the opinion expressed by Airy and others, the wave becomes less steep in front and steeper behind.

If, now, we take account of the fact that, as may easily be inferred from Equation 5.1, the wave-surface becomes steeper in proportion as p is increased, we are then justified in saying that a solitary wave which is steeper than the stationary one, corresponding to the same height, becomes less steep in front and steeper *behind*, but that its behaviour is exactly opposite when it is *less steep* than the stationary one.

5.2. Cnoidal Waves

Applying Equation 2.12 to the cnoidal wave,

$$\eta = h \cdot \mathrm{cn}^2\, px\,, \tag{5.4}$$

---

[1] The left side of the figure is the front side of the wave, because the wave has been made stationary by the application of a positive velocity (i.e. from left to right) to the fluid.

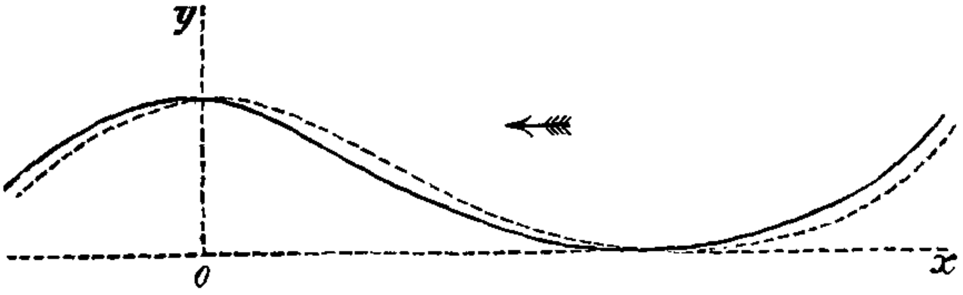

Figure 5.2.: Change of wave-form calculated for $\left(h - 4\sigma\,\mathbf{M}^2\,p^2 > 0\right)$

we get

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = -\frac{3q_0ph}{l} \cdot \left\{ \frac{2 \cdot \left[\alpha - \sigma p^2 \cdot \left(2 - 4\,\mathbf{M}^2\right)\right]}{3 \cdot \left(4\sigma\,\mathbf{M}^2\,p^2 - h\right)} - \mathrm{cn}^2\,px \right\} \cdot \left(4\sigma\,\mathbf{M}^2\,p^2 - h\right) \cdot \mathrm{sn}\,px \cdot \mathrm{cn}\,px \cdot \mathrm{dn}\,px\,. \tag{5.5}$$

Supposing then

$$2 \cdot \left[\alpha - \sigma p^2 \cdot \left(2 - 4\,\mathbf{M}^2\right)\right] = 3 \cdot \left(4\sigma\,\mathbf{M}^2\,p^2 - h\right)\ ,$$

we have

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = -\frac{3q_0ph}{l} \cdot \left(4\sigma\,\mathbf{M}^2\,p^2 - h\right) \cdot \mathrm{sn}^3\,px \cdot \mathrm{cn}\,px \cdot \mathrm{dn}\,px \tag{5.6}$$

Here Figure 5.2 shows the change of form calculated for the case $h - 4\sigma\,\mathbf{M}^2\,p^2 > 0$. When $h - 4\sigma\,\mathbf{M}^2\,p^2 = 0$, the waves are stationary in accordance with Equation 3.7, whilst for $h - 4\sigma\,\mathbf{M}^2\,p^2 < 0$ they become steeper *behind*; and this last result, since p is inversely proportional to the wave-length, may be stated by saying that cnoidal waves become less steep in front and steeper behind when, for a given modulus and a given height, their length is smaller than the one required for the stationary wave of this modulus and height. In proportion as $\mathbf{M}$ is taken smaller the cnoidal waves more and more resemble sinusoidal waves. They would take the sinusoidal form for $\mathbf{M} = 0$, but then an infinitely small wavelength would be required for the stationary case. For this reason sinusoidal waves may always be considered as cnoidal waves whose length is too large to be stationary, that is, they are always becoming steeper in front.

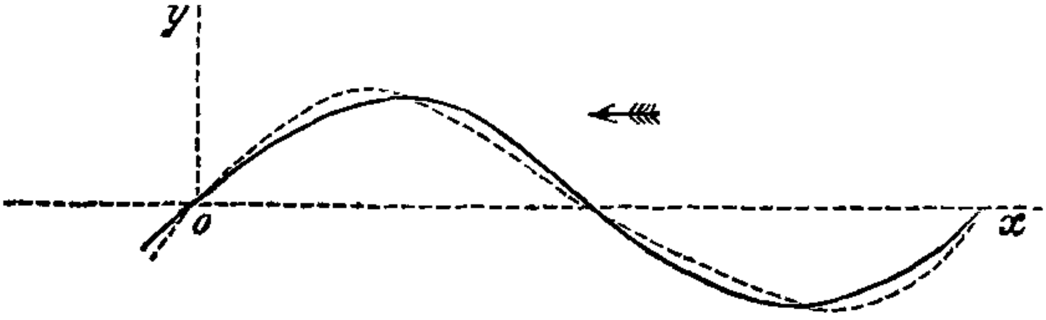

Figure 5.3.: Change of wave-form calculated for sinusoidal waves

5.3. Sinusoidal Waves

This last result is easily verified by direct application of Equation 2.12 to the equation of a train of sinusoidal waves:

$$\eta = A \cdot \sin\left(\frac{2\pi x}{\lambda}\right) ;$$

for, supposing

$$\alpha = \frac{2\pi^2\sigma}{\lambda^2} ,$$

we obtain

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{3q_0\pi A^2}{2l\lambda} \cdot \sin\left(\frac{4\pi x}{\lambda}\right) ;$$

and from this the change of form indicated in Figure 5.3 is easily calculated.

5.4. More complicated Cases

For the sake of curiosity, we represent by means of the following figures the change of form for some more complicated cases.

Figure 5.4 and Figure 5.5 refer to the equation

$$\eta = A_1 \cdot \sin\left(\frac{2\pi x}{\lambda}\right) + \frac{1}{3}A_1 \cdot \sin\left(\frac{4\pi x}{\lambda}\right) .$$

In Figure 5.4, $\frac{A_1}{l}$ is supposed to be small compared with $\left(\frac{l}{\lambda}\right)^2$, as is the case with waves of extremely small height. In Figure 5.5 we suppose $\left(\frac{l}{\lambda}\right)^2$ to be small in regard to $\frac{A_1}{l}$. Generally for more complicated forms of waves these two cases have to be discriminated. When there is a moderate proportionality between the two fractions, the result is still more complicated.

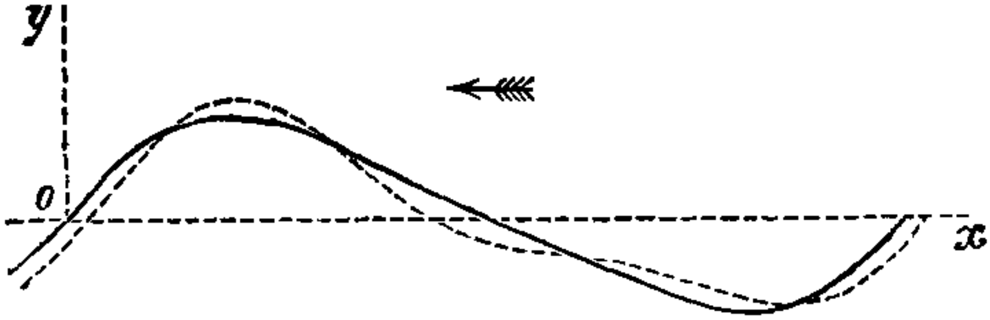

Figure 5.4.: Sinusoidal wave – more complicated case

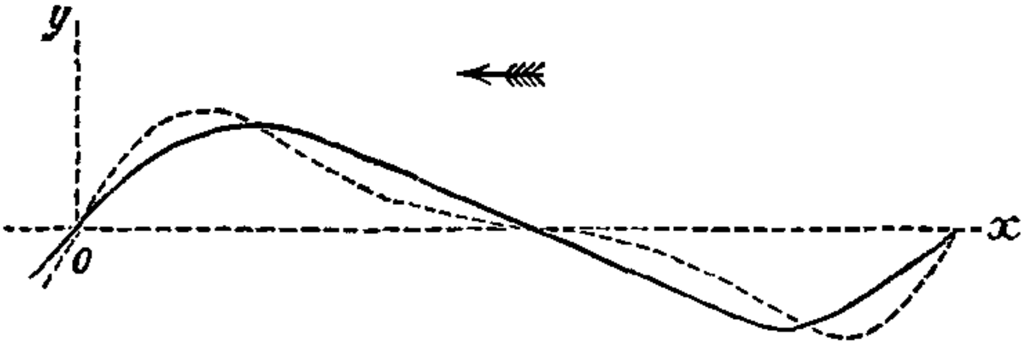

Figure 5.5.: Sinusoidal wave – more complicated case

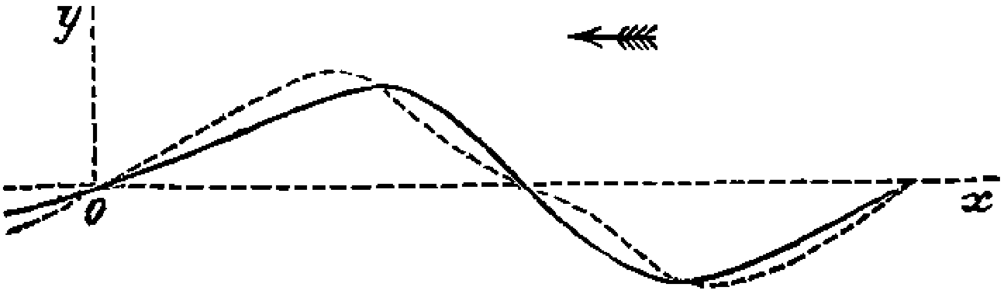

Figure 5.6.: Sinusoidal wave – more complicated case

Finally, Figure 5.6 refers to the equation

$$\eta = A_1 \cdot \sin\left(\frac{2\pi x}{\lambda}\right) - \frac{1}{3} A_1 \cdot \sin\left(\frac{4\pi x}{\lambda}\right) ,$$

in case that $\left(\frac{l}{\lambda}\right)^2$ is the smaller fraction.
It is worthy of remark that all these waves grow steeper in front.

6. Calculation of the Fluid Motion for Stationary Waves to the Higher Order of Approximation

In order to remove every doubt as to the existence of absolutely stationary waves, we will show how by development in rapidly convergent series the state of motion of the fluid belonging to such a wave-motion may be calculated. Expressing again the horizontal and vertical velocity of a particle by means of the series given in Equation 2.1 and Equation 2.2, which fulfil all the conditions for the interior of the fluid, we have only, neglecting capillarity, to satisfy the surface-conditions

$$v_1 = u_1 \cdot \frac{\partial \eta}{\partial x} \tag{6.1}$$

and

$$u_1^2 + v_1^2 + 2g\eta = \text{const.} \tag{6.2}$$

For the case of cnoidal waves, which is the general one, we have found as a first approximation

$$\left(\frac{\partial \eta}{\partial x}\right)^2 = \frac{3}{l^3}\eta \cdot (h - \eta) \cdot (k + \eta) \, .$$

But now, to obtain higher approximations, we assume, indicating by accents differentiation with respect to x:

$$\eta'^2 = a\eta \cdot (h - \eta) \cdot (k + \eta) \cdot \left(1 + b\eta + c\eta^2 + \cdots\right) \, , \tag{6.3}$$

and

$$f = q + r\eta + s\eta^2 + t\eta^3 + u\eta^4 + \cdots \tag{6.4}$$

On writing out Equation 6.3, neglecting such terms as are of a higher order than the fourth compared with η, h, and k, which latter quantities are of the same order, we obtain

$$\eta'^2 = ahk\eta + \{a \cdot (h - k) + abhk\} \cdot \eta^2 + \{-a + ab \cdot (h - k)\} \cdot \eta^3 - ab\eta^4 \tag{6.5}$$

and by differentiation

$$\eta'' = \frac{1}{2}ahk + \{a\cdot(h-k) + abhk\}\cdot\eta + \left\{-\frac{3}{2}a + \frac{3}{2}ab\cdot(h-k)\right\}\eta^2 - 2ab\eta^3\,. \tag{6.6}$$

From Equation 6.4, by successive differentiations and substitutions, retaining all terms up to the third and the $3\frac{1}{2}^{\text{th}}$ order, we deduce:

$$\begin{aligned}
f' &= \left(r + 2s\eta + 3t\eta^2\right)\cdot\eta'\,;\\
f'' &= \frac{1}{2}arhk + \{ar\cdot(h-k) + abrhk + 3ashk\}\cdot\eta\\
&\quad + \left\{-\frac{3}{2}ar + \frac{3}{2}abr\cdot(h-k) + 4as\cdot(h-k)\right\}\cdot\eta^2 + \{-2abr - 5as\}\cdot\eta^3\,;\\
f''' &= [ar\cdot(h-k) + abrhk + 3ashk\\
&\quad + \{-3ar + 3abr\cdot(h-k) + 8as\cdot(h-k)\}\cdot\eta + \{-6abr - 15as\}\cdot\eta^2]\cdot\eta'\,;\\
f^{iv} &= \frac{1}{2}a^2rh\cdot(h-k) + \left\{a^2r\cdot(h-k)^2 - \frac{3}{2}a^2rhk\right\}\cdot\eta\\
&\quad - \frac{15}{2}a^2r\cdot\{h-k\}\cdot\eta^2 + \frac{15}{2}a^2r\eta^3\,;\\
f^{v} &= \left[a^2r\cdot(h-k)^2 - \frac{3}{2}a^2rhk - 15a^2r\cdot\{h-k\}\cdot\eta + \frac{45}{2}a^2r\eta^2\right]\cdot\eta'\,;
\end{aligned}$$

where η' is a quantity of the order of $\frac{3}{2}$.
Substituting these values in Equation 2.11, where $y = l + \eta$, we have, retaining terms of the third order:

$$\begin{aligned}
u_1 &= f - \frac{1}{2}l^2\cdot f'' - l\eta\cdot f'' + \frac{1}{24}l^4\cdot f^{iv}\\
&= q - \frac{1}{4}arl^2hk + \frac{1}{48}a^2rl^4hk\cdot(h-k)\\
&\quad + \left\{r - \frac{1}{2}arl^2\cdot(h-k) - \frac{1}{2}abrl^2hk - \frac{3}{2}asl^2hk - \frac{1}{2}arlhk\right.\\
&\quad \left. + \frac{1}{24}a^2rl^4\cdot(h-k)^2 - \frac{3}{16}a^2rl^4hk\right\}\cdot\eta\\
&\quad + \left\{s + \frac{3}{4}al^2r - \frac{3}{4}abrl^2\cdot(h-k) - 2asl^2\cdot(h-k)\right.\\
&\quad \left. - arl\cdot(h-k) - \frac{5}{16}a^2rl^4\cdot(h-k)\right\}\cdot\eta^2\\
&\quad + \left\{t + abrl^2 + \frac{5}{2}asl^2 + \frac{3}{2}arl + \frac{5}{16}a^2rl^4\right\}\cdot\eta^3\,.
\end{aligned} \tag{6.7}$$

We find in the same way, including terms of the $3\frac{1}{2}^{\text{th}}$ order:

$$
\begin{aligned}
v_1 &= -l \cdot f' - \eta \cdot f' + \frac{1}{6} l^3 \cdot f''' + \frac{1}{2} l^2 \eta \cdot f''' - \frac{1}{120} l^5 \cdot f^v \\
&= \eta' \Bigg[-rl + \frac{1}{6} arl^3 \cdot (h-k) + \frac{1}{6} abrl^3 hk + \frac{1}{2} asl^3 hk - \frac{1}{120} a^2 rl^5 \cdot (h-k)^2 \\
&\quad + \frac{3}{80} a^2 rl^5 hk + \Bigg\{ -2sl - r - \frac{1}{2} arl^3 + \frac{1}{2} abrl^3 \cdot (h-k) \\
&\quad + \frac{4}{3} asl^3 \cdot (h-k) + \frac{1}{2} arl^2 \cdot (h-k) + \frac{1}{8} a^2 rl^5 \cdot (h-k) \Bigg\} \cdot \eta \\
&\quad \Bigg\{ -3tl - 2s - abrl^3 - \frac{5}{2} asl^3 - \frac{3}{2} arl^2 - \frac{3}{16} a^2 rl^5 \Bigg\} \cdot \eta^2 \Bigg]
\end{aligned}
\tag{6.8}
$$

If now we write, in accordance with Equation 6.1

$$
u_1 = \frac{v_1}{\eta'} = A + B\eta + C\eta^2 + D\eta^3 + \cdots ,
\tag{6.9}
$$

we have from Equation 6.7 and Equation 6.8:

$$
\begin{aligned}
A &= q - \frac{1}{4} arl^2 hk + \frac{1}{48} a^2 rl^4 hk \cdot (h-k) \\
&= -rl + \frac{1}{6} arl^3 \cdot (h-k) \\
&\quad + \frac{1}{6} abrl^3 hk + \frac{1}{2} asl^3 hk - \frac{1}{120} a^2 rl^5 \cdot (h-k)^2 + \frac{3}{80} a^2 rl^5 hk
\end{aligned}
\tag{6.10}
$$

$$
\begin{aligned}
B &= r - \frac{1}{2} arl^2 \cdot (h-k) - \frac{1}{2} abrl^2 hk - \frac{3}{2} asl^2 hk - \frac{1}{2} arlhk \\
&\quad + \frac{1}{24} a^2 rl^4 \cdot (h-k)^2 - \frac{3}{16} a^2 rl^4 hk \\
&= -2sl - r - \frac{1}{2} arl^3 + \frac{1}{2} abrl^3 \cdot (h-k) \\
&\quad + \frac{4}{3} asl^3 \cdot (h-k) + \frac{1}{2} arl^2 \cdot (h-k) + \frac{1}{8} a^2 rl^5 \cdot (h-k)
\end{aligned}
\tag{6.11}
$$

$$
\begin{aligned}
C &= s + \frac{3}{4} al^2 r - \frac{3}{4} abrl^2 \cdot (h-k) - 2asl^2 \cdot (h-k) \\
&\quad - arl \cdot (h-k) - \frac{5}{16} a^2 rl^4 \cdot (h-k) \\
&= -3tl - 2s - abrl^3 - \frac{5}{2} asl^3 - \frac{3}{2} arl^2 - \frac{3}{16} a^2 rl^5
\end{aligned}
\tag{6.12}
$$

$$
D = t + abrl^2 + \frac{5}{2} asl^2 + \frac{3}{2} arl + \frac{5}{16} a^2 rl^4 .
\tag{6.13}
$$

Moreover, since Equation 6.2 may be written in the form

$$u_1^2 \cdot \left(1 + \eta'^2\right) + 2g\eta = \left(A + B\eta + C\eta^2 + D\eta^3\right)^2 \cdot \left(1 + ahk\eta \right. \\ \left. + a \cdot (h-k) \cdot \eta^2 - a\eta^3\right) + 2g\eta = \text{const.} \tag{6.14}$$

we readily obtain

$$2AB + ahkA^2 + 2g = 0\,, \tag{6.15}$$
$$2AC + B^2 + a \cdot (h-k) \cdot A^2 = 0\,, \tag{6.16}$$
$$2AD + 2BC - aA^2 = 0\,. \tag{6.17}$$

From Equation 6.10, Equation 6.11, Equation 6.12, Equation 6.15, Equation 6.16, Equation 6.17, the six quantities q, r, s, t, a, and b may be calculated, and if we had retained everywhere terms of one higher order, we might have got eight equations with eight unknown quantities, and so forth.

By a first approximation we readily obtain from Equation 6.10 till Equation 6.13:

$$\begin{aligned} r &= -\frac{q}{l} \\ s &= \frac{q}{l^2} + \frac{1}{4}aql \\ t &= -\frac{q}{l^3} - \frac{1}{3}aq + \frac{1}{3}abql - \frac{7}{48}a^2ql^3 \\ A &= q \\ B &= -\frac{q}{l} \\ C &= \frac{q}{l^2} - \frac{1}{2}aql \\ D &= -\frac{q}{l^3} + \frac{2}{3}aq - \frac{2}{3}abql + \frac{1}{6}a^2ql^3 \end{aligned}$$

and then from Equation 6.15 till Equation 6.17:

$$\begin{aligned} q^2 &= gl \\ a &= \frac{3}{l^3} \\ b &= \frac{3}{4l} \end{aligned}$$

Proceeding to the second approximation, we find

$$\begin{aligned}
r &= -\frac{q}{l}\cdot\left(1+\frac{(h-k)}{2l}\right)\\
s &= \frac{q}{l^2}+\frac{1}{4}aql+\frac{7}{4}\frac{q}{l^2}\cdot\left(\frac{(h-k)}{l}\right)\\
A &= q\\
B &= -\frac{q}{l}+\frac{q}{l}\cdot\left(\frac{(h-k)}{l}\right)\\
C &= \frac{q}{l^2}-\frac{1}{2}aql-\frac{19}{8}\frac{q}{l^2}\cdot\left(\frac{(h-k)}{l}\right)
\end{aligned}$$

and then again from Equation 6.15 and Equation 6.16:

$$\begin{aligned}
q^2 &= gl\cdot\left(1+\frac{(h-k)}{l}\right)\\
a &= \frac{3}{l^3}-\frac{15}{4}\cdot\left(\frac{(h-k)}{l^4}\right)
\end{aligned}$$

Finally, a third approximation leads to:

$$\begin{aligned}
r &= -\frac{q}{l}\cdot\left(1+\frac{(h-k)}{2l}-\frac{9}{20}\cdot\frac{(h-k)^2}{l^2}-\frac{93}{80}\cdot\frac{hk}{l^2}\right)\\
A &= q+\frac{3}{4}q\cdot\frac{hk}{l^2}\\
B &= -\frac{q}{l}+\frac{q}{l}\cdot\frac{(h-k)}{l}-\frac{21}{20}\frac{q}{l}\cdot\frac{(h-k)^2}{l^2}-\frac{12}{5}\frac{q}{l}\cdot\frac{hk}{l^2}\\
q^2 &= gl\cdot\left(1+\frac{(h-k)}{l}-\frac{1}{20}\cdot\frac{(h-k)^2}{l^2}-\frac{33}{20}\cdot\frac{hk}{l^2}\right)
\end{aligned}$$

By means of these results we may now readily obtain from Equation 2.1 and Equation 2.2 expressions for u and v including respectively the terms of the 2^{nd} and $2\frac{1}{2}^{\text{th}}$ order.
They are:

$$\begin{aligned}
u = \sqrt{gl}\cdot\Bigg\{&\left(1+\frac{(h-k)}{2l}-\frac{3}{20}\cdot\frac{(h-k)^2}{l^2}-\frac{33}{40}\cdot\frac{hk}{l^2}\right)\\
&+\left(1+\frac{(h-k)}{l}\right)\cdot\frac{\eta}{l}+\frac{7}{4}\frac{\eta^2}{l^2}\\
&+\left(\frac{3}{4}\cdot\frac{hk}{l^2}+\frac{3}{2}\cdot\frac{(h-k)}{l}\frac{\eta}{l}-\frac{9}{4}\cdot\frac{\eta^2}{l^2}\right)\cdot y^2\Bigg\}
\end{aligned}\tag{6.18}$$

$$v = \eta' \sqrt{\frac{g}{l}} \cdot \left\{ \left(1 + \frac{(h-k)}{l} - \frac{7}{4} \cdot \frac{\eta}{l} \right) \cdot y + \left(-\frac{(h-k)}{2l} + \frac{3}{2} \cdot \frac{\eta}{l} \right) \cdot y^3 \right\} , \qquad (6.19)$$

where

$$\eta'^2 = \frac{3}{l^3} \cdot \left(1 - \frac{5}{4} \cdot \frac{(h-k)}{l} \right) \cdot (h-\eta) \cdot (k+\eta) \cdot \left(1 + \frac{3}{4} \cdot \frac{\eta}{l} \right) . \qquad (6.20)$$

7. Calculation of the Equation of the Surface

We will now show how for the equation of the surface of a stationary train of waves a more correct expression than Equation 3.7 can be deduced. For this purpose, we have to integrate the differential equation Equation 6.3, or rather we have to prove that a series can be given which solves this equation to any desired degree of accuracy. Now such a series may be obtained in the following manner. Let

$$\eta_1 = h_1 \cdot \operatorname{cn}^2 \frac{1}{2} x \cdot \sqrt{a \cdot (h_1 + k_1)} \quad \left(\operatorname{mod} \mathbf{M} = \sqrt{\frac{h_1}{h_1 + k_1}} \right) \tag{7.1}$$

represent the solution of an equation

$$\eta_1'^2 = a\eta_1 \cdot (h_1 - \eta_1) \cdot (k_1 + \eta_1) \tag{7.2}$$

where h_1 and k_1 have values which are slightly different from those of h and k in Equation 6.3; then these values and the coefficients α, β, &c., of a series

$$\eta = \alpha\eta_1 + \beta\eta_1^2 + \gamma\eta_1^3 + \delta\eta_1^4 + \cdots \tag{7.3}$$

may be determined in such a way[1] that this series Equation 7.3 satisfies the equation Equation 6.3.
Indeed, substituting Equation 7.3 in Equation 6.3 and taking into account Equation 7.2, Equation 6.3 reduces to

$$\begin{aligned} &\left(\alpha + 2\beta\eta_1 + 3\gamma\eta_1^2 + \cdots\right)^2 \cdot (h_1 - \eta_1) \cdot (k_1 + \eta_1) \\ =& \left(\alpha + \beta\eta_1 + \gamma\eta_1^2 + \cdots\right) \cdot \left(h - \alpha\eta_1 - \beta\eta_1^2 - \gamma\eta_1^3 + \cdots\right) \\ &\cdot \left(k + \alpha\eta_1 + \beta\eta_1^2 + \gamma\eta_1^3 + \cdots\right) \cdot \left(1 + b\alpha\eta_1 + (b\beta + c\alpha) \cdot \eta_1^2 + \cdots\right) , \end{aligned}$$

and it is only necessary to equalize the coefficients of the corresponding terms of both members of this equation.

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[1] The coefficient a in Equation 7.2 might also have been chosen slightly different in value from a in Equation 6.3, but this would only have introduced an unnecessary indeterminateness in the solution.

If we retain all terms to the fourth order, we find in this way, after some reductions:

$$\alpha h_1 k_1 - hk = 0 \tag{7.4}$$
$$\alpha^2 \cdot (h_1 - k_1) - \alpha^2 \cdot (h - k) - \left(b\alpha^2 - 3\beta\right) \cdot hk = 0 \tag{7.5}$$
$$-\alpha^3 + \alpha^4 - \left(b\alpha^4 - 2\alpha^2\beta\right) \cdot (h - k) - \left(c\alpha^3 - 2b\alpha^2\beta + 8\beta^2 - 5\alpha\gamma\right) \cdot hk = 0 \tag{7.6}$$
$$-4\alpha\beta + 3\alpha^2\beta + b\alpha^4 - \left(c\alpha^3 + 3b\alpha^2\beta - 3\beta^2 - 4\alpha\gamma\right) \cdot (h - k) = 0 \tag{7.7}$$
$$-4\beta^2 - 6\alpha\gamma + c\alpha^4 + 4b\alpha^3\beta + 3\alpha\beta^2 + 3\alpha^2\gamma = 0\,. \tag{7.8}$$

To a *first* approximation these equations are satisfied by taking

$$\left.\begin{aligned} h_1 &= h \\ k_1 &= k \\ \alpha &= 1 \\ \beta &= b \\ \gamma &= b^2 + \frac{1}{3}c\,. \end{aligned}\right\} \tag{7.9}$$

If then we substitute in Equation 7.4, Equation 7.5, Equation 7.6, and Equation 7.7

$$\begin{aligned} h_1 &= h + \epsilon \\ k_1 &= k + \text{æ} \\ \alpha &= 1 + \alpha_1 \\ \beta &= b + \beta_1\,, \end{aligned}$$

where α and β are quantities of the first, ϵ and æ of the second order, we find from these equations by *second* approximation:

$$\left.\begin{aligned} \epsilon &= -bhk \\ \text{æ} &= bhk \\ \alpha_1 &= -b \cdot (h - k) \\ \beta_1 &= \left(-2b^2 + \frac{1}{3}c\right) \cdot (h - k)\,. \end{aligned}\right\} \tag{7.10}$$

Substituting as a *third* approximation:

$$\begin{aligned} h_1 &= h - bhk + \epsilon_1 \\ k_1 &= k + bhk + \text{æ}_1 \\ \alpha &= 1 - b \cdot (h - k) + \alpha_2 \end{aligned}$$

we obtain finally:

$$\left.\begin{aligned}\epsilon_1 &= \frac{1}{3}chk\cdot(-h+2k)\\ \text{æ}_1 &= \frac{1}{3}chk\cdot(2h-k)\\ \alpha_2 &= \left(b^2-\frac{2}{3}c\right)\cdot\left(h^2-hk+k^2\right)\ .\end{aligned}\right\} \tag{7.11}$$

Hence, the equation of the surface of the waves is, including all terms of the third order:

$$\begin{aligned}\eta = &\left[1-b\cdot(h-k)+\left(b^2-\frac{2}{3}c\right)\cdot\left(h^2-hk+k^2\right)\right]\cdot\eta_1\\ &+\left[b+\left(-2b^2+\frac{1}{3}c\right)\cdot(h-k)\right]\cdot\eta_1^2+\left[b^2+\frac{1}{3}c\right]\cdot\eta_1^3+\cdots\end{aligned} \tag{7.12}$$

where

$$\eta_1 = h_1\cdot\mathrm{cn}^2\,\frac{1}{2}x\cdot\sqrt{a\cdot(h_1+k_1)}\quad\left(\mathrm{mod}\,\mathbf{M}=\sqrt{\frac{h_1}{h_1+k_1}}\right) \tag{7.1}$$

and

$$\left.\begin{aligned}h_1 &= h-bhk+\frac{1}{3}chk\cdot(-h+2k)\\ k_1 &= k+bhk+\frac{1}{3}chk\cdot(2h-k)\,.\end{aligned}\right\} \tag{7.13}$$

Here, according to Equation 6.20,

$$\left.\begin{aligned}a &= \frac{3}{l^3}\cdot\left(1-\frac{5}{4}\cdot\frac{(h-k)}{l}+\cdots\right)\\ b &= \frac{3}{4l}+\cdots\ ,\end{aligned}\right\} \tag{7.14}$$

whereas the value of c and more correct expressions for a and b could only have been obtained by means of still more tedious calculations, which we have not executed. If we confine ourselves to that degree of approximation for which all the calculations have been effected, we may write for the equation of the wave-surface:

$$\eta = \left[1-\frac{3\cdot(h-k)}{4l}\right]\cdot\eta_1+\frac{3}{4l}\cdot\eta_1^2 \tag{7.15}$$

$$\eta_1 = h\cdot\mathrm{cn}^2\,\frac{1}{2}\cdot\left(1-\frac{5\cdot(h-k)}{8l}\right)\cdot x\cdot\sqrt{\frac{3\cdot(h+k)}{l^3}} \tag{7.16}$$

$$\mathbf{M} = \left(1 - \frac{3k}{8l}\right) \cdot \sqrt{\frac{h}{h+k}}\,. \tag{7.17}$$

For the solitary wave, when $k = 0$, we have[2]

$$\eta = \left[1 - \frac{3h}{8l}\right] \cdot \eta_1 + \frac{3}{4l} \cdot \eta_1^2 \tag{7.18}$$

$$\eta_1 = h \cdot \operatorname{sech}^2 \frac{1}{2} \cdot \left(1 - \frac{5h}{8l}\right) \cdot x \cdot \sqrt{\frac{3h}{l^3}}\,. \tag{7.19}$$

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[2] Another close approximation of the surface-equation of this wave has been deduced by McCowan, [7], page 48.

A. The Korteweg–De Vries Equation in Modern Notation

In modern notation the KORTEWEG–DE VRIES equation reads

$$\partial_t u + 6\, u\, \partial_x u + \partial_x^3 u = 0\,. \tag{A.1}$$

The KdV equation is a nonlinear, dispersive partial differential equation for a function u of two dimensionless real variables, x and t, which are proportional to space and time respectively. ∂_x and ∂_t denote partial derivatives with respect to x and t.
It can be derived from Equation 2.12

$$\frac{\partial \eta}{\partial t} = \frac{3}{2} \cdot \sqrt{\frac{g}{l}} \frac{\partial}{\partial x} \left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma \cdot \frac{\partial^2 \eta}{\partial x^2} \right) \tag{2.12}$$

through the following substitutions:

$$t' \equiv \frac{1}{2}\sqrt{\frac{g}{l\sigma}} \cdot t \tag{A.2a}$$

$$x' \equiv -\frac{x}{\sqrt{\sigma}} \tag{A.2b}$$

$$u \equiv \frac{1}{2}\eta + \frac{1}{3}\alpha\,. \tag{A.2c}$$

Differentiation of Equation A.2a with respect to ∂t and Equation A.2b with respect to ∂x yields

$$\partial t = \frac{2 \cdot \partial t'}{\sqrt{\frac{g}{l\sigma}}} \tag{A.3a}$$

$$\partial x = -\sqrt{\sigma} \cdot \partial x'\,. \tag{A.3b}$$

Equation A.2c is rearranged to give

$$\eta = 2u - \frac{2}{3}\alpha\,. \tag{A.4}$$

From Equation A.3b it follows that

$$\partial x^2 = \sigma \cdot \partial x'^2 \,. \tag{A.5}$$

Inserting all this in Equation 2.12 yields

$$\begin{aligned}\frac{\partial\left(2u-\frac{2}{3}\alpha\right)}{2\cdot\partial t'}\cdot\sqrt{\frac{g}{l\sigma}} = \frac{3}{2}\sqrt{\frac{g}{l}}\cdot\frac{\partial}{-\sqrt{\sigma}\cdot\partial x'}&\left(\frac{1}{2}\left(2u-\frac{2}{3}\alpha\right)^2\right.\\ &\left.+\frac{2}{3}\alpha\cdot\left(2u-\frac{2}{3}\alpha\right)+\frac{1}{3}\sigma\cdot\frac{\partial^2\left(2u-\frac{2}{3}\alpha\right)}{\sigma\cdot\partial x'^2}\right)\,.\end{aligned} \tag{A.6}$$

Cancelling terms gives

$$\begin{aligned}\frac{\partial\left(2u-\frac{2}{3}\alpha\right)}{\partial t'} = -\frac{\partial}{\partial x'}&\left(\frac{3}{2}\left(2u-\frac{2}{3}\alpha\right)^2\right.\\ &\left.+2\alpha\cdot\left(2u-\frac{2}{3}\alpha\right)+\frac{\partial^2\left(2u-\frac{2}{3}\alpha\right)}{\partial x'^2}\right)\,.\end{aligned} \tag{A.7}$$

Differentiation of the left term in Equation A.7 gives:

$$\frac{\partial\left(2u-\frac{2}{3}\alpha\right)}{\partial t'} = \frac{\partial(2u)}{\partial t'} + \frac{\partial\left(-\frac{2}{3}\alpha\right)}{\partial t'} = 2\cdot\frac{\partial u}{\partial t'}\,. \tag{A.8}$$

Differentiation of the right term in brackets on the r.h.s. of Equation A.7 gives:

$$\frac{\partial^2\left(2u-\frac{2}{3}\alpha\right)}{\partial x'^2} = \frac{\partial^2(2u)}{\partial x'^2} + \frac{\partial^2\left(-\frac{2}{3}\alpha\right)}{\partial x'^2} = 2\cdot\frac{\partial^2 u}{\partial x'^2}\,. \tag{A.9}$$

The two remaining terms in brackets on the r.h.s. of Equation A.7 are multiplied to give

$$\begin{aligned}\frac{3}{2}\left(2u-\frac{2}{3}\alpha\right)^2+2\alpha\cdot\left(2u-\frac{2}{3}\alpha\right) &= \frac{3}{2}\left(4u^2-\frac{8}{3}u\alpha+\frac{4}{9}\alpha^2\right)\\ &\quad+4u\alpha-\frac{4}{3}\alpha^2\\ &= 6u^2-4u\alpha+\frac{2}{3}\alpha^2+4u\alpha-\frac{4}{3}\alpha^2\\ &= 6u^2-\frac{2}{3}\alpha^2\,.\end{aligned} \tag{A.10}$$

Inserting Equation A.8, Equation A.9 and Equation A.10 in Equation A.7 yields

$$2 \cdot \frac{\partial u}{\partial t'} = -\frac{\partial}{\partial x'} \left(6u^2 - \frac{2}{3}\alpha^2 + 2 \cdot \frac{\partial^2 u}{\partial x'^2} \right) \tag{A.11}$$

and differentiating again gives

$$\begin{aligned} 2 \cdot \frac{\partial u}{\partial t'} &= -\frac{\partial \left(6u^2\right)}{\partial x'} + \frac{\partial \left(-\frac{2}{3}\alpha^2\right)}{\partial x'} - 2 \cdot \frac{\partial^3 u}{\partial x'^3} \\ 2 \cdot \frac{\partial u}{\partial t'} &= -12u \cdot \frac{\partial u}{\partial x'} - 2 \cdot \frac{\partial^3 u}{\partial x'^3} \,. \end{aligned} \tag{A.12}$$

Multiplying both sides of Equation A.12 with $\frac{1}{2}$ finally gives

$$\frac{\partial u}{\partial t'} = -6u \frac{\partial u}{\partial x'} - \frac{\partial^3 u}{\partial x'^3} \tag{A.13}$$

or

$$\frac{\partial u}{\partial t'} + 6u \frac{\partial u}{\partial x'} + \frac{\partial^3 u}{\partial x'^3} = 0 \,. \tag{A.1}$$

B. Jacobi Elliptic Functions and Elliptic Integrals

The JACOBI elliptic functions[1] are *standard forms* of elliptic functions. The three basic functions are denoted $\mathrm{cn(u,k)}$ (*cosinus amplitudinis*), $\mathrm{dn(u,k)}$ (*delta amplitudinis*), and $\mathrm{sn(u,k)}$ (*sinus amplitudinis*)[2], where k is known as the *elliptic modulus*. From these three basic functions the remaining 9 functions are derived. In sum there are 12 JACOBI elliptic functions.
The name elliptic function or integral stems from the fact that the perimeter of an ellipse is most simply expressed in terms of the complete LEGENDRE elliptic integral of the second kind, and this is why the functions are so called.

$\int R(w,x)\,\mathrm{d}x$ and $\int R(\sqrt{f(x)},x)\,\mathrm{d}x$ respectively are a general form of an elliptic integral, where $R(w,x)$ is a rational function of x and w, w^2 is a function of x, i.e. $f(x)=w^2$ or $\sqrt{f(x)}=w$ respectively and $f(x)$ is a polynomial function that is cubic or quartic in x, and $R(w,x)$ contains at least one odd power of w, and w^2 has no repeated factors. Furthermore elliptic integrals can be viewed as generalizations of the inverse trigonometric functions, known as *elliptic functions*, but these cannot, in general, be expressed in terms of elementary functions.
However, with the appropriate reduction formula, every elliptic integral can be brought into a form that involves integrals over rational functions $P(w,x), Q(w,x)$ and the three LEGENDRE canonical forms, i.e. the *incomplete* elliptic integrals of the first, second and third kind denoted by $F(\phi,k)$, $E(\phi,k)$, and $\Pi(n;\phi,k)$, respectively. In case of $\phi=\pi/2$, then the integrals are called *complete* elliptic integrals and are denoted $K(k)$, $E(k)$, and $\Pi(n;k)$.

JACOBI elliptic functions arise from the inversion of the *incomplete* elliptic integral u of the first kind (type I),

$$u = F(\phi,k) = \int_0^{\phi} \frac{\mathrm{d}t}{\sqrt{1-k^2\cdot\sin^2 t}}\,, \tag{B.1}$$

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[1] Taken mainly from https://mathworld.wolfram.com/, https://dlmf.nist.gov/22 and Wikipedia

[2] This notation was introduced by CHRISTOPH GUDERMANN (* 25. March 1798, Vienenburg; † 25. September 1851, Münster) in 1838. The terminology stems from the old, and outdated, notation of JACOBI: $\mathrm{sn}\,u = \sin\mathrm{am}\,u$, $\mathrm{cn}\,u = \cos\mathrm{am}\,u$, $\mathrm{dn}\,u = \Delta\,\mathrm{am}\,u$

where $0 < k^2 < 1,\ k = \operatorname{mod} u$ is the elliptic modulus, and $\phi = \operatorname{am}(u, k) = \operatorname{am}(u)$, with $-\pi/2 < \phi < \pi/2$ is the JACOBI amplitude, an infinitely-valued function of u, giving

$$\phi = \frac{1}{F}(u, k) = \operatorname{am}(u, k)\,. \tag{B.2}$$

From Equation B.2 the relationships

$$\begin{aligned}
\sin(\phi) &= \sin(\operatorname{am}(u, k)) \\
&= \operatorname{sn}(u, k) \\
\cos(\phi) &= \cos(\operatorname{am}(u, k)) \\
&= \operatorname{cn}(u, k) \\
\sqrt{1 - k^2 \cdot \sin^2(\phi)} &= \sqrt{1 - k^2 \cdot \sin^2(\operatorname{am}(u, k))} \\
&= \operatorname{dn}(u, k)\,.
\end{aligned}$$

follow.

Equation B.1 is the trigonometric form of the elliptic integral type I; substituting $\theta = \sin t$ and $x = \sin \phi$, one obtains the LEGENDRE normal form:[3]

$$F(x; k) = \int_0^x \frac{\mathrm{d}\theta}{\sqrt{(1 - \theta^2)\,(1 - k^2\theta^2)}}\,. \tag{B.3}$$

The *complete* elliptic integral of the first kind is given by $K(k) = F\left(\frac{\pi}{2}, k\right)$, or

$$K(k) = \int_0^{\pi/2} \frac{\mathrm{d}t}{\sqrt{1 - k^2 \cdot \sin^2 t}} = \int_0^1 \frac{\mathrm{d}\theta}{\sqrt{(1 - \theta^2) \cdot (1 - k^2\theta^2)}}\,, \tag{B.4}$$

if one substitutes $x = \sin(\pi/2) = 1$ and $\sin t = \theta$.

In general the integrals with a lower integral limit of 0 are called *incomplete* elliptic integrals. Is also the upper integral limit $\frac{\pi}{2}$, in the case of types I and II we speak of complete elliptic integrals.

Incomplete elliptic integrals of the second kind E (type II) in trigonometric form are defined as

$$E(\phi, k) = \int_0^{\phi} \sqrt{1 - k^2 \cdot \sin^2 t}\,\mathrm{d}t\,. \tag{B.5}$$

Substituting $\theta = \sin t$ and $x = \sin \phi$, one obtains the LEGENDRE normal form:

$$E(x, k) = \int_0^x \frac{\sqrt{1 - k^2\theta^2}}{\sqrt{1 - \theta^2}}\,\mathrm{d}\theta\,. \tag{B.6}$$

---

[3] $\sin t = \theta$, $\Rightarrow \frac{\mathrm{d}\theta}{\mathrm{d}t} = \cos t$, and therefore $\mathrm{d}t = \frac{\mathrm{d}\theta}{\cos t}$, which gives, recognizing $\sin^2 t + \cos^2 t = 1$ $\frac{\mathrm{d}\theta}{\sqrt{1-\theta^2}}$.

Complete elliptic integrals of the second kind E in trigonometric and LEGENDRE normal form are defined as

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \cdot \sin^2 t}\, \mathrm{d}t = \int_0^1 \frac{\sqrt{1 - k^2\theta^2}}{\sqrt{1 - \theta^2}}\, \mathrm{d}\theta \tag{B.7}$$

with $\sin(\pi/2) = 1$ and $\theta = \sin t$.

Incomplete elliptic integrals of the third kind Π in trigonometric form are defined as

$$\Pi(n; \phi,\ \alpha) = \int_0^{\phi} \frac{1}{(1 - n\sin^2 t)} \frac{\mathrm{d}t}{\sqrt{1 - (\sin t \sin \alpha)^2}}\,. \tag{B.8}$$

The number n is called the *characteristic* and can take on any value, independently of the other arguments. Substituting $\theta = \sin t$ and $x = \sin \phi$, one obtains the LEGENDRE normal form:

$$\Pi(n; \phi, k) = \int_0^{\sin\phi} \frac{1}{(1 - n\theta^2)} \frac{\mathrm{d}\theta}{\sqrt{(1 - k^2\theta^2) \cdot (1 - \theta^2)}}\,. \tag{B.9}$$

Complete elliptic integrals of the third kind Π in trigonometric and LEGENDRE normal form are defined as

$$\Pi(n; \phi, k) = \int_0^{\phi/2} \frac{1}{(1 - n\sin^2 t)} \frac{\mathrm{d}t}{\sqrt{1 - k^2 \cdot \sin^2 t}} \tag{B.10}$$

$$\Pi(n; \phi, k) = \int_0^{1} \frac{1}{(1 - n\theta^2)} \frac{\mathrm{d}\theta}{\sqrt{(1 - k^2\theta^2) \cdot (1 - \theta^2)}} \tag{B.11}$$

with $\sin(\pi/2) = 1$ and $\theta = \sin t$.

The functions $\mathrm{sn}(u,k)$, $\mathrm{cn}(u,k)$, and $\mathrm{dn}(u,k)$ are doubly periodic generalizations of the trigonometric functions satisfying, for the limiting cases $k=0$ and $k=1$, the following relations:

function	$k=0$	$k=1$
$\mathrm{sn(u,k)}$	$\sin(u)$	$\tanh(u)$
$\mathrm{cn(u,k)}$	$\cos(u)$	$\mathrm{sech}(u)=\frac{1}{\cosh(z)}$
$\mathrm{dn(u,k)}$	1	$\mathrm{sech}(u)=\frac{1}{\cosh(u)}$

Table B.1.: Limiting forms of JACOBIAN elliptic functions with regard to the elliptic modulus k

I.e., When $k=0$ or $k=1$ the JACOBI elliptic functions degenerate into trigonometric or hyperbolic functions.

The three main JACOBI elliptic functions $\mathrm{sn}(u)$, $\mathrm{cn}(u)$, and $\mathrm{dn}(u)$, from which all others are derived are periodic in $K(k)$ and $K'(k)$.
In general, the double periodicity of the JACOBI elliptic functions may be expressed as:

$$\mathrm{pq}(u+2\alpha K(k^2)+2i\beta K(1-k^2)\,,\,k^2)=(-1)^\gamma\,\mathrm{pq}(u,k^2) \tag{B.12}$$

where α and β are any pair of integers. $K(\cdot)$ is the complete elliptic integral of the first kind, also known as the quarter period. The power of negative unity (γ) is given in the following table:

		q			
		c	s	n	d
p	c	0	β	$\alpha+\beta$	α
	s	β	0	α	$\alpha+\beta$
	n	$\alpha+\beta$	α	0	β
	d	α	$\alpha+\beta$	β	0

Table B.2.: Values for the power γ of negative unity

When the factor $(-1)^\gamma$ is equal to -1, the equation expresses quasi-periodicity. When it is equal to unity, it expresses full periodicity. It can be seen, for example, that for the entries containing only α when α is even, full periodicity is expressed by the above equation, and the function has full periods of $4K(k^2)$ and $2\,\mathrm{i}\,K(1-k^2)$. Likewise, functions with entries containing only β have full periods of $2K(k^2)$ and $4\,\mathrm{i}\,K(1-k^2)$, while those with $\alpha+\beta$ have full periods of $4K(k^2)$ and $4\,\mathrm{i}\,K(1-k^2)$. For the three main JACOBI elliptic functions, along with their zeros and poles, the result is shown in Table B.3.

function	periods	zero	pole
$\mathrm{sn}(u,k)$	$4K, 2\,\mathrm{i}\,K'$	$2mK + 2n\,\mathrm{i}\,K'$	$2mK + (2n+1)\,\mathrm{i}\,K'$
$\mathrm{cn}(u,k)$	$4K, 2(K+\mathrm{i}\,K')$	$(2m+1)K + 2n\,\mathrm{i}\,K'$	$2mK + (2n+1)\,\mathrm{i}\,K'$
$\mathrm{dn}(u,k)$	$2K, 4\,\mathrm{i}\,K'$	$(2m+1)K + (2n+1)\,\mathrm{i}\,K'$	$2mK + (2n+1)\,\mathrm{i}\,K'$
$n, m \in \mathbb{Z}$			

Table B.3.: Periodicity, zeros and poles of $\mathrm{sn}(u)$, $\mathrm{cn}(u)$, and $\mathrm{dn}(u)$

$K(k)$ is the *complete elliptic integral of the first kind* as stated above with $K'(k) = K(k')$, $k' = \sqrt{1-k^2}$ (the *complementary modulus*) and therefore $K(k)$ and $K'(k)$ are interrelated through the elliptic integrals

$$K = K(k) = \int_0^{\pi/2} \frac{\mathrm{d}\,t}{\sqrt{1-k^2\sin^2 t}}, \quad K'(k) = \int_0^{\pi/2} \frac{\mathrm{d}\,t}{\sqrt{1-(1-k^2)\cdot\sin^2 t}}\,.$$

E.g. $\mathrm{sn}(u,k)$ has zeros at $u=0$ and $u=2K$ and the poles at $u=\mathrm{i}\,K'$ and $u = 2K + \mathrm{i}\,K'$.

The $\mathrm{cn}\,x$, $\mathrm{dn}\,x$, and $\mathrm{sn}\,x$ functions may also be defined as solutions to the second-order nonlinear ordinary differential equations

$$\begin{aligned}
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \left(2-k^2\right)y - 2y^3 \\
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= -\left(1-2k^2\right)y - 2k^2y^3 \\
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= -\left(1+k^2\right)y + 2k^2y^3
\end{aligned}$$

respectively.

The standard JACOBI elliptic functions satisfy the identities

$$\begin{aligned} \operatorname{sn}^2 u + \operatorname{cn}^2 u &= 1 \\ k^2 \operatorname{sn}^2 u + \operatorname{dn}^2 u &= 1 \\ k^2 \operatorname{cn}^2 u + k'^2 &= \operatorname{dn}^2 u \\ \operatorname{cn}^2 u + k'^2 \operatorname{sn}^2 u &= \operatorname{dn}^2 u \,. \end{aligned}$$

Special values include

$$\begin{aligned} \operatorname{cn}(0,k) &= \operatorname{cn}(0) = 1 \\ \operatorname{cn}(K(k),k) &= \operatorname{cn}(K(k)) = 0 \\ \operatorname{dn}(0,k) &= \operatorname{dn}(0) = 1 \\ \operatorname{dn}(K(k),k) &= \operatorname{dn}(K(k)) = k' = \sqrt{1-k^2} \\ \operatorname{sn}(0,k) &= \operatorname{sn}(0) = 1 \\ \operatorname{sn}(K(k),k) &= \operatorname{sn}(K(k)) = 0 \end{aligned}$$

where $K = K(k)$ is a complete elliptic integral of the first kind and $k' = \sqrt{1-k^2}$ is the complementary elliptic modulus, and

$$\operatorname{cn}(u,1) = \operatorname{sech} u \tag{B.13}$$

$$\operatorname{dn}(u,1) = \operatorname{sech} u \tag{B.14}$$

$$\operatorname{sn}(u,1) = \tanh u \,, \tag{B.15}$$

as stated above.

In terms of integrals,

$$
\begin{aligned}
u &= \int_{0}^{\mathrm{sn}(u,k)} \left(1-t^2\right)^{-1/2} \left(1-k^2t^2\right)^{-1/2} \mathrm{d}t \\
&= \int_{\mathrm{ns}(u,k)}^{\infty} \left(t^2-1\right)^{-1/2} \left(t^2-l^2\right)^{-1/2} \mathrm{d}t \\
&= \int_{\mathrm{cn}(u,k)}^{1} \left(1-t^2\right)^{-1/2} \left(k'^2+k^2t^2\right)^{-1/2} \mathrm{d}t \\
&= \int_{1}^{\mathrm{nc}(u,k)} \left(t^2-1\right)^{-1/2} \left(k'^2t^2+k^2\right)^{-1/2} \mathrm{d}t \\
&= \int_{\mathrm{dn}(u,k)}^{1} \left(1-t^2\right)^{-1/2} \left(t^2-k'^2\right)^{-1/2} \mathrm{d}t \\
&= \int_{1}^{\mathrm{nd}(u,k)} \left(t^2-1\right)^{-1/2} \left(1-k'^2t^2\right)^{-1/2} \mathrm{d}t \\
&= \int_{0}^{\mathrm{sc}(u,k)} \left(1+t^2\right)^{-1/2} \left(1+k'^2t^2\right)^{-1/2} \mathrm{d}t \\
&= \int_{\mathrm{cs}(u,k)}^{\infty} \left(t^2+1\right)^{-1/2} \left(t^2+k'^2\right)^{-1/2} \mathrm{d}t \\
&= \int_{0}^{\mathrm{sd}(u,k)} \left(1-k'^2t^2\right)^{-1/2} \left(1+k^2t^2\right)^{-1/2} \mathrm{d}t \\
&= \int_{\mathrm{ds}(u,k)}^{\infty} \left(t^2-k'^2\right)^{-1/2} \left(t^2+k^2\right)^{-1/2} \mathrm{d}t \\
&= \int_{1}^{\mathrm{cd}(u,k)} \left(1-t^2\right)^{-1/2} \left(1-k^2t^2\right)^{-1/2} \mathrm{d}t \\
&= \int_{\mathrm{dc}(u,k)}^{1} \left(t^2-1\right)^{-1/2} \left(t^2-k^2\right)^{-1/2} \mathrm{d}t
\end{aligned}
$$

Jacobi elliptic functions addition formulas include (where, for example, $\mathrm{sn}(u,k)$ is written as $\mathrm{sn}\,u$ for conciseness)

$$
\begin{aligned}
\mathrm{sn}(u+v) &= \frac{\mathrm{sn}\,u\,\mathrm{cn}\,v\,\mathrm{dn}\,v + \mathrm{sn}\,v\,\mathrm{cn}\,u\,\mathrm{dn}\,u}{1-k^2\,\mathrm{sn}^2\,u\,\mathrm{sn}^2\,v} \\
\mathrm{cn}(u+v) &= \frac{\mathrm{cn}\,u\,\mathrm{cn}\,v - \mathrm{sn}\,u\,\mathrm{sn}\,v\,\mathrm{dn}\,u\,\mathrm{dn}\,v}{1-k^2\,\mathrm{sn}^2\,u\,\mathrm{sn}^2\,v} \\
\mathrm{dn}(u+v) &= \frac{\mathrm{dn}\,u\,\mathrm{dn}\,v - k^2\,\mathrm{sn}\,u\,\mathrm{sn}\,v\,\mathrm{cn}\,u\,\mathrm{cn}\,v}{1-k^2\,\mathrm{sn}^2\,u\,\mathrm{sn}^2\,v}
\end{aligned}
$$

Extended to integral periods,

$$
\begin{aligned}
\operatorname{sn}(u+K) &= \frac{\operatorname{cn} u}{\operatorname{dn} u} \\
\operatorname{cn}(u+K) &= -\frac{k' \operatorname{sn} u}{\operatorname{dn} u} \\
\operatorname{dn}(u+K) &= \frac{k'}{\operatorname{dn} u} \\
\operatorname{sn}(u+2K) &= -\operatorname{sn} u \\
\operatorname{cn}(u+2K) &= -\operatorname{cn} u \\
\operatorname{dn}(u+2K) &= \operatorname{dn} u
\end{aligned}
$$

For complex arguments,

$$
\begin{aligned}
\operatorname{sn}(u+\mathrm{i}\,v) &= \frac{\operatorname{sn}(u,k)\operatorname{dn}(v,k')}{1-\operatorname{dn}^2(u,k)\operatorname{sn}^2(v,k')} \\
&\quad + \mathrm{i}\left(\frac{\operatorname{cn}(u,k)\operatorname{dn}(u,k)\operatorname{sn}(v,k')\operatorname{cn}(v,k')}{1-\operatorname{dn}^2(u,k)\operatorname{sn}^2(v,k')}\right) \\
\operatorname{cn}(u+\mathrm{i}\,v) &= \frac{\operatorname{cn}(u,k)\operatorname{cn}(v,k')}{1-\operatorname{dn}^2(u,k)\operatorname{sn}^2(v,k')} \\
&\quad - \mathrm{i}\left(\frac{\operatorname{sn}(u,k)\operatorname{dn}(u,k)\operatorname{sn}(v,k')\operatorname{dn}(v,k')}{1-\operatorname{dn}^2(u,k)\operatorname{sn}^2(v,k')}\right) \\
\operatorname{dn}(u+\mathrm{i}\,v) &= \frac{\operatorname{dn}(u,k)\operatorname{cn}(v,k')\operatorname{dn}(v,k')}{1-\operatorname{dn}^2(u,k)\operatorname{sn}^2(v,k')} - \mathrm{i}\left(\frac{k^2\operatorname{sn}(u,k)\operatorname{cn}(u,k)\operatorname{sn}(v,k')}{1-\operatorname{dn}^2(u,k)\operatorname{sn}^2(v,k')}\right)
\end{aligned}
$$

Derivatives of the Jacobi elliptic functions include

$$
\begin{aligned}
\frac{\mathrm{d}(\operatorname{sn} u)}{\mathrm{d}u} &= \operatorname{cn} u \operatorname{dn} u \\
\frac{\mathrm{d}(\operatorname{cn} u)}{\mathrm{d}u} &= -\operatorname{sn} u \operatorname{dn} u \\
\frac{\mathrm{d}(\operatorname{dn} u)}{\mathrm{d}u} &= -k^2 \operatorname{sn} u \operatorname{cn} u
\end{aligned}
$$

Double-period formulas involving the Jacobi elliptic functions include

$$
\begin{aligned}
\operatorname{sn}(2u) &= \frac{2\operatorname{sn} u \operatorname{cn} u \operatorname{dn} u}{1-k^2\operatorname{sn}^4 u} \\
\operatorname{cn}(2u) &= \frac{1-2\operatorname{sn}^2 u + k^2 \operatorname{sn}^4 u}{1-k^2\operatorname{sn}^4 u} \\
\operatorname{dn}(2u) &= \frac{1-2k^2\operatorname{sn} u + k^2 \operatorname{sn}^4 u}{1-k^2\operatorname{sn}^4 u}\,.
\end{aligned}
$$

Half-period formulas involving the Jacobi elliptic functions include

$$\begin{aligned}
\operatorname{sn}\left(\frac{1}{2}K\right) &= \frac{1}{\sqrt{1+k'}} \\
\operatorname{cn}\left(\frac{1}{2}K\right) &= \sqrt{\frac{k'}{1+k'}} \\
\operatorname{dn}\left(\frac{1}{2}K\right) &= \sqrt{k'}\,.
\end{aligned}$$

Squared formulas include

$$\begin{aligned}
\operatorname{sn}^2 u &= \frac{1-\operatorname{cn}(2u)}{1+\operatorname{dn}(2u)} \\
\operatorname{cn}^2 u &= \frac{\operatorname{dn}(2u)+\operatorname{cn}(2u)}{1+\operatorname{dn}(2u)} \\
\operatorname{dn}^2 u &= \frac{\operatorname{dn}(2u)+\operatorname{cn}(2u)}{1+\operatorname{cn}(2u)}\,.
\end{aligned}$$

Taylor series of the Jacobi elliptic functions were considered by Hermite (1863), Schett, [11], [10] and Dumont [5],

$$\begin{aligned}
\operatorname{cn}(u,k) &= 1-\frac{1}{2}u^2+\frac{1}{24}\cdot\left(1+4k^2\right)u^4-\frac{1}{720}\cdot\left(1+44k^2+16k^4\right)u^6+\cdots \\
\operatorname{dn}(u,k) &= 1-\frac{1}{2}k^2u^2+\frac{1}{24}\cdot\left(4k^2+k^4\right)u^4-\frac{1}{720}\cdot\left(16k^2+44k^4+k^6\right)u^6+\cdots \\
\operatorname{sn}(u,k) &= u-\frac{1}{6}\cdot\left(1+k^2\right)u^3+\frac{1}{120}\cdot\left(1+14k^2+k^4\right)u^5+\cdots\,.
\end{aligned}$$

Bibliography

[1] George Biddell Airy. Tides and waves. In Hugh James Rose, editor, *Encyclopædia Metropolitana. Mixed Sciences*, volume 5, pages 241–396. 1841.

[2] A. B. Basset. *An Elementary Treatise on Hydrodynamics and Sound.* Cambridge: Deighton, Bell, and Co. London: George Bell and Sons, 1 edition, 1890.

[3] J. Boussinesq. Theorie de i'intumescence liquid, appleteonde solitaire au de translation, se propageantdansun canal rectangulaire. *Les Comptes Rendus de l'Académie des Sciences*, 72:755–759, 1871.

[4] A. Cayley. *An elementary treatise on elliptic functions.* Cambridge: Deighton, Bell, and Co. London: George Bell and Sons, 1876.

[5] D. Dumont. Une approche combinatoire des fonctions elliptiques de jacobi. *Advances in Mathematics*, 41:1–39, 1981.

[6] H. Lamb. *A Treatise on the Motion of Fluids.* Cambridge: Deighton, Bell, and Co. London: George Bell and Sons, 1 edition, 1879.

[7] J. McCowan. On the solitary wave. *Philosophical Magazine*, 32(194):45–58, 1891.

[8] Lord Rayleigh. On waves. *Philosophical Magazine*, 1(4):257–279, 1876.

[9] A. J. C. B. Saint-Venant de. Mouvements des molécules de l'onde dite solitaire, propagée à la surface de l'eau d'un canal. *Les Comptes Rendus de l'Académie des Sciences*, 101:1101–1105, 1215–1218, 1445–1447, 1885.

[10] A. Schett. Properties of the taylor series expansion coefficients of the jacobian elliptic functions. *Mathematics of Computation*, 30(133):143–147, 1976. `https://www.ams.org/journals/mcom/1976-30-133/S0025-5718-1976-0391477-3/S0025-5718-1976-0391477-3.pdf`.

[11] A. Schett. Recurrence formula of the taylor series expansion coefficients of the jacobian elliptic functions. *Mathematics of Computation*, 31(140):1003–1005, 1977. `https://www.ams.org/journals/mcom/1977-31-140/S0025-5718-1977-0442301-2/S0025-5718-1977-0442301-2.pdf`.

[12] G. G. Stokes. On the theory of oscillatory waves. In G. G. Stokes, editor, *Mathematical and Physical Papers*, volume 1, pages 197–229. Cambridge: Deighton, Bell, and Co. London: George Bell and Sons, 1841.

[13] G. G. Stokes. On the theory of oscillatory waves. *Transactions of the Cambridge Philosophical Society*, 8:441–455, 1847.